The derivation of the basic Black-Scholes options pricing equation follows from imposing the condition that a riskless portfolio made up of stock and options must return the same interest rate as other riskless assets, assuming stock and options prices are in a market equilibrium. The portfolio will have an options component and a variable quantity of stock so that it remains riskless at all positive stock prices. That leads to a relationship between the option price $V(S,t)$ and fixed parameters for the interest rate r, the stock's volatility σ, and the stock's price $S(t)$, which is known up to t but not afterward.

The derivation takes a fixed risk-free interest rate r and the stock's volatility σ to be known constants through time T when any options will expire. It is assumed that there are no transactions costs or constraints, no taxes, no dividends, and no liquidity constraints. Trading in securities is in continuous units and instantaneous; price changes are completely unaffected by the trader under consideration. It is assumed that no risk-free arbitrage opportunities exist. (These assumptions are listed in Hull (1997), p 236.)

Assume stock prices evolve according to the stochastic process called geometric Brownian motion:

$$dS = \mu S dt + \sigma S \varphi \sqrt{dt}$$

where $\varphi \sim N(0,1)$

This is an Ito process. The assumption can be justified by evidence but we will not do it here. The first time is a "drift" in the mean over time. If μ is .01, then in time dt the stock's mean rise is 1%. The second term is a volatility term with mean zero.

Consider a portfolio made up of N shares of stock and short one option (a put or call, not specified yet). Let $V(S,t)$ denote the value of the option and $\Pi(S,t)$ denote the value of the portfolio. Then

$$\Pi(S,t) = NS(t) - V(S,t) \qquad \text{at each time } t$$

Let N be a real-valued choice variable. Let it be chosen at every time t so as to make the portfolio riskless – meaning that $d\Pi$ has no stochastic component.

$$d\Pi = NdS - dV$$

There is no dN term because N is a control variable not a source of independent change.

Use Ito's Lemma (a rule for calculating differentials of quantities dependent on stochastic processes) to evaluate dV:

$$dV = \left(\mu S \tfrac{\partial V}{\partial S} + \tfrac{\partial V}{\partial t} + \tfrac{\sigma^2 S^2}{2} \tfrac{\partial^2 V}{\partial S^2}\right)dt + \sigma S \tfrac{\partial V}{\partial S} \varphi \sqrt{dt}$$

So

$$d\Pi = NdS - dV$$
$$= N[\mu S dt + \sigma S\varphi\sqrt{dt}] - dV$$
$$= N[\mu S dt + \sigma S\varphi\sqrt{dt}] - [(\mu S\frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2})dt + \sigma S\frac{\partial V}{\partial S}\varphi\sqrt{dt}]$$

Grouping the stochastic terms together gives:

$$= N[\mu S dt - (\mu S\frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2})]dt + [N\sigma S - \sigma S\frac{\partial V}{\partial S}]\varphi\sqrt{dt}$$

The goal was to choose N to as to make the stochastic component equal to zero. We can accomplish that by choosing N to satisfy

$$N\sigma S - \sigma S\frac{\partial V}{\partial S} = 0$$
$$\Rightarrow \quad N = \frac{\partial V}{\partial S}$$

That simplifies $d\Pi$ to:

$$d\Pi = -[\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2}]dt$$

Because the portfolio is riskless, it must in a market equilibrium return the known fixed risk-free interest rate r. If it returned some other interest rate, traders could make infinite profits in arbitrage borrowing at the lower rate and investing at the higher rate.

$$d\Pi = r\Pi dt$$
$$= r[S\frac{\partial V}{\partial S} - V]dt$$

Equate the last two equations for $d\Pi$:

$$d\Pi = -[\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2}]dt = r[S\frac{\partial V}{\partial S} - V]dt$$
$$\Rightarrow \quad -\frac{\partial V}{\partial t} - \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} = r[S\frac{\partial V}{\partial S} - V]$$

which takes us to the Black-Scholes Equation:

$$\boxed{\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0}$$

This equation is a parabolic PDE. It relates the option price $V(S,t)$ to parameters for a fixed risk-free interest rate r, the stock's fixed volatility σ, and the stock's price $S(t)$.

The pricing equation can be used to compute the value of any collection of securities with a value at time T that is known in terms of $S(T)$. A call on the stock, for example, pays off zero if $S(T)$ is less than the call's strike price, and pays off the difference between the $S(T)$ and the strike price if the stock price is higher. Computation of the equations for an option's price are detailed operations, not solved here. Mathematically, the price of the option at terminal time T acts as a boundary condition on the partial differential equation.

References

Hull, John C. 1997. *Options, Futures, and Other Derivatives*, third edition. Prentice Hall.

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